Evolving Notes by Felix Otto for ISTA summer school, version July 28th 2022

These are evolving notes; they present selected aspects of the work arXiv:2112.10739 (with P. Linares, M. Tempelmayr, and P. Tsatsoulis) with additional motivation. For a simpler setting, we also recommend to have a look at arXiv:2207.10627 (with P. Linares). The algebraic aspects are worked out in arXiv:2103.04187 (with P. Linares and M. Tempelmayr). Thanks to Markus Tempelmayr and Kihoon Seong for proofreading.

1. A singular quasi-linear SPDE

We are interested in nonlinear elliptic or parabolic equations with a random and thus typically rough right hand side ξ. Our goal is to move beyond the well-studied semi-linear case. We consider a mildly quasi-linear case where the coefficients of the leading-order derivatives depend on the solution u itself. To fix ideas, we focus on the parabolic case in a single space dimension; since we treat the parabolic equation in the whole space-time like an anisotropic elliptic equation, we denote by x_1 the space-like and by x_2 the time-like variable. Hence we propose to consider

ao22 (1) $$(\partial_2-\partial_1^2)u=a(u)\partial_1^2u+\xi,$$

where we think of the values a_0 of $a(u)$ to be so small such that $\partial_2-a_0\partial_1^2$ is parabolic. We are interested in laws / ensembles of ξ where the solutions v to the linear equation

ao25 (2) $$(\partial_2-\partial_1^2)v=\xi$$

are (almost surely) Hölder continuous with exponent $\alpha\in(0,1)$. In view of the parabolic nature, Hölder continuity is measured w. r. t. the Carnot-Carathéodory distance

ao79 (3) $$\text{“}|y-x|\text{''}:=|y_1-x_1|+\sqrt{|y_2-x_2|}.$$

By Schauder theory for $\partial_2-\partial_1^2$, which we shall expand on below, this is the case when ξ is in the (negative) Hölder space $C^{\alpha-2}$. We note that this range includes white noise ξ, since the latter is in $C^{-\frac{D}{2}-}$, where D is the effective (space-time) dimension, which in our parabolic case is $D=1+2=3$, see Subsection 2 for more details.

In the range of $\alpha\in(0,1)$, the SPDE (1) is what is “singular”: We cannot expect the product $a(u)\partial_1^2u$ to be canonically defined. Indeed, at least for smooth a, we may hope for $a(u)\in C^{\alpha}$, but we cannot hope for more than $\partial_1^2u\in C^{\alpha-2}$. Hence for $\alpha<1$, the function $a(u)$ is less regular then the distribution ∂_1^2u is irregular.

The same feature occurs for the (semi-linear) multiplicative heat equation $(\partial_2-\partial_1^2)u=a(u)\xi$; in fact, our approach also applies to this semi-linear case, which already has been treated by (standard) regularity structures in Hairer-Pardoux '15. A singular product is already present in the case when the x_1-dependence is suppressed, so that the above semi-linear equation turns into the SDE $\frac{du}{dx_2} = a(u)\xi$ with white noise ξ in the time-like variable x_2. In this case, the analogue of v from (2) is Brownian motion, which is known to be Hölder continuous with exponent $\frac{1}{2}-$ in x_2, which in view of (3) corresponds to the border-line setting $\alpha = 1-$. Ito's integral and, more recently, rough paths (Lyons) and controlled rough path (Gubinelli) have been devised to tackle the issue in this setting.

sec:Schauder

2. Annealed Schauder theory

This section provides the main (linear) PDE ingredient for our result. At the same time, it will allow us to discuss (3).

In view of (3), we are interested in the fundamental solution of the differential operator $A := \partial_2 - \partial_1^2$. It turns out to be convenient to use the more symmetric[1] fundamental solution of $A^*A = (-\partial_2 - \partial_1^2)(\partial_2 - \partial_1^2)$ $= \partial_1^4 - \partial_2^2$. Moreover, it will be more transparent to "disintegrate" the latter fundamental solution, by which we mean writing it as $\int_0^\infty dt\psi_t(z)$, where $\{\psi_t\}_{t>0}$ are the kernels of the semi-group $\exp(-tA^*A)$ generated by the non-negative operator A^*A. Clearly, the Fourier transform $\mathcal{F}\psi_t(q)$ is given by $= \exp(-t(q_1^4 + q_2^2))$; in particular, ψ_t is a Schwartz function. For a Schwartz distribution f like realizations of white noise, we thus define $f_t(y)$ as the pairing of f with $\psi_t(y - \cdot)$; f_t is a smooth function. On the level of these kernels, the semi-group property translates into

ao36 (4)
$$\psi_s * \psi_t = \psi_{s+t} \quad \text{and} \quad \int \psi_t = 1.$$

By scale invariance under $x_1 = \lambda\hat{x}_1$, $x_2 = \lambda^2\hat{x}_2$, and $t = \lambda^4\hat{t}$, we have

ao37 (5)
$$\psi_t(x_1, x_2) = \frac{1}{(\sqrt[4]{t})^{D=3}}\psi_1\Big(\frac{x_1}{\sqrt[4]{t}}, \frac{x_2}{(\sqrt[4]{t})^2}\Big).$$

By construction, ψ satisfies the PDE

ao80 (6)
$$\partial_t\psi_t + (\partial_1^4 - \partial_2^2)\psi_t = 0.$$

lem:int **Lemma 1.** *Let $0 < \alpha \le \eta < \infty$ with $\eta \notin \mathbb{Z}$, $p < \infty$, and $x \in \mathbb{R}^2$ be given. For a random Schwartz distribution f with*

ao76 (7)
$$\mathbb{E}^{\frac{1}{p}}|f_t(y)|^p \le (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t} + |y - x|)^{\eta-\alpha} \quad \textit{for all } t > 0, y \in \mathbb{R}^2,$$

there exists a unique random function u of the class

ao55 (8)
$$\sup_{y\in\mathbb{R}^2} \frac{1}{|y - x|^\eta}\mathbb{E}^{\frac{1}{p}}|u(y)|^p < \infty$$

[1]it is symmetric under reflection space and time

satisfying (distributionally in $\mathbb{R}^2$*)*

ao56 (9)
$$(\partial_2 - \partial_1^2)u = f + \textit{polynomial of degree} \le \eta - 2.$$

It actually satisfies (9) without the polynomial. Moreover, the l. h. s. of (8) is bounded by a constant only depending on α *and* η*.*

Now white noise ξ is an example of such a random Schwartz distribution: Since $\xi_t(y)$ is a centered Gaussian, we have $\mathbb{E}^{\frac{1}{p}}|\xi_t(y)|^p \lesssim_p \mathbb{E}^{\frac{1}{2}}(\xi_t(y))^2$. By the characterizing property of white noise, we have $\mathbb{E}^{\frac{1}{2}}(\xi_t(y))^2 = \big(\int \psi_t^2(y-\cdot)\big)^{\frac{1}{2}}$, which by scaling (5) is equal to

$$\Big(\frac{1}{\sqrt[4]{t}}\Big)^{\frac{D}{2}}\Big(\int \psi_1^2\Big)^{\frac{1}{2}} \sim \Big(\frac{1}{\sqrt[4]{t}}\Big)^{\frac{3}{2}},$$

which can be interpreted as stating that in an annealed sense, ξ is in the Hölder class $C^{-\frac{D}{2}}$. Hence the assumptions of Lemma 1 are satisfied with $\alpha = \eta = \frac{1}{2}$.

Fixing a "base-point" x, Lemma 1 thus constructs the solution of (2) distinguished by $v(x) = 0$. Note that the output (8) takes the form of $\mathbb{E}^{\frac{1}{p}}|v(y)-v(x)|^p \lesssim_p |y-x|^{\frac{1}{2}}$, which amounts to a Hölder continuity condition, centered in x, and in an annealed sense. Hence Lemma 1 provides an annealed version of a Schauder estimate, alongside a Liouville-type uniqueness result.

Proof of Lemma 1 By construction, $\int_0^\infty dt(-\partial_2 - \partial_1^2)\psi_t$ is the fundamental solution of $\partial_2 - \partial_1^2$, so that we take the convolution of it with f. However, in order to obtain a convergent expression for $t \uparrow \infty$, we need to pass to a Taylor remainder:

ao74 (10)
$$u = \int_0^\infty dt(\mathrm{id} - \mathrm{T}_x^\eta)(-\partial_2 - \partial_1^2)f_t,$$

where T_x^η the operation of taking the Taylor polynomial of order $\le \eta$; as we shall argue the additional Taylor polynomial does not affect the PDE. We claim that (10) is well-defined and estimated as

$$\mathbb{E}^{\frac{1}{p}}|u(y)|^p \lesssim |y-x|^\eta.$$

To this purpose, we first note that

ao77 (11)
$$\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}} f_t(y)|^p \lesssim (\sqrt[4]{t})^{\alpha-2-|\mathbf{n}|}(\sqrt[4]{t} + |y-x|)^{\eta-\alpha},$$

where

ao26 (12)
$$\partial^{\mathbf{n}} u := \partial_1^{n_1}\partial_2^{n_2} u \quad \text{and} \quad |\mathbf{n}| = n_1 + 2n_2.$$

Indeed, by the semi-group property (4) we may write $\partial^{\mathbf{n}} f_t(y) = \int dz\, \partial^{\mathbf{n}}\psi_{\frac{t}{2}}(y-z)\, f_{\frac{t}{2}}(z)$, so that $\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}} f_t(y)|^p \le \int dz |\partial^{\mathbf{n}}\psi_{\frac{t}{2}}(y-z)| \mathbb{E}^{\frac{1}{p}}|f_{\frac{t}{2}}(z)|^p$. Hence by (7), (11) follows from the kernel bound $\int dz\ |\partial^{\mathbf{n}}\psi_{\frac{t}{2}}(y-z)|$

$(\sqrt[4]{t}+|y-x|)^{\eta-\alpha} \lesssim (\sqrt[4]{t})^{-|\mathbf{n}|}(\sqrt[4]{t}+|y-x|)^{\eta-\alpha}$, which itself is a consequence of the scaling (5) and the fact that $\psi_{\frac{1}{2}}$ is a Schwartz function.

Equipped with (11), we now derive two estimates for the integrand of (10), namely for $\sqrt[4]{t} \geq |y-x|$ ("far field") and for $\sqrt[4]{t} \leq |y-x|$ ("near field"). We write the Taylor remainder $(\mathrm{id}-\mathrm{T}_x^\eta)(\partial_2+\partial_1^2)f_t(y)$ as a linear combination of[2] $(y-x)^{\mathbf{n}}\partial^{\mathbf{n}}(\partial_2+\partial_1^2)f_t(z)$ with $|\mathbf{n}|>\eta$ and at some point z intermediate to y and x. By (11) such a term is estimated by $|y-x|^{|\mathbf{n}|}(\sqrt[4]{t})^{\alpha-4-|\mathbf{n}|}(\sqrt[4]{t}+|y-x|)^{\eta-\alpha}$, which in the far field is $\sim |y-x|^{|\mathbf{n}|}(\sqrt[4]{t})^{\eta-4-|\mathbf{n}|}$. Since the exponent on t is <-1, we obtain as desired

$$\mathbb{E}^{\frac{1}{p}}\Big|\int_{|y-x|^4}^\infty dt(\mathrm{id}-\mathrm{T}_x^\eta)(\partial_2+\partial_1^2)f_t(y)\Big|^p \lesssim |y-x|^\eta.$$

For the near-field term, i. e. for $\sqrt[4]{t} \leq |y-x|$, we proceed as

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{p}}|(\mathrm{id}-\mathrm{T}_x^\eta)(\partial_2+\partial_1^2)f_t(y)|^p \\
&\leq \mathbb{E}^{\frac{1}{p}}|(\partial_2+\partial_1^2)f_t(y)|^p + \sum_{|\mathbf{n}|\leq\eta}|y-x|^{|\mathbf{n}|}\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}}(\partial_2+\partial_1^2)f_t(x)|^p \\
&\overset{(11)}{\lesssim} (\sqrt[4]{t})^{\alpha-4}|y-x|^{\eta-\alpha} + \sum_{|\mathbf{n}|\leq\eta}|y-x|^{|\mathbf{n}|}(\sqrt[4]{t})^{\eta-4-|\mathbf{n}|}.
\end{aligned}$$

Since η is not an integer, the sum restricts to $|\mathbf{n}|<\eta$, so that all exponents on t are >-1. Hence we obtain as desired

$$\mathbb{E}^{\frac{1}{p}}\Big|\int_0^{|y-x|^4} dt(\mathrm{id}-\mathrm{T}_x^\eta)(\partial_2+\partial_1^2)f_t(y)\Big|^p \lesssim |y-x|^\eta.$$

We return to the discussion of the singular product, in its simplest form of

$$v\partial_1^2 v = \partial_1^2\frac{1}{2}v^2 - (\partial_1 v)^2.$$

While in view of Lemma 1 the first r. h. s. term is well-defined as a Schwartz distribution, we now argue that the second term diverges. Since it has a sign, it diverges as a distribution iff it diverges as a function; hence it is enough to argue that its pointwise expectation diverges. Indeed, applying ∂_1 to the representation formula (10), so that the constant Taylor term drops out, we have

$$\partial_1 v = \int_0^\infty dt\partial_1(-\partial_2-\partial_1^2)\xi_t. \tag{13}$$

[2]where $x^{\mathbf{n}} := x_1^{n_1}x_2^{n_2}$

We note that for the integrand

ao81

$$\begin{aligned} &\mathbb{E}^{\frac{1}{2}}(\partial_1(-\partial_2-\partial_1^2)\xi_t(y))^2 \\ &= \Big(\int(\partial_1(-\partial_2-\partial_1^2)\psi_t)^2\Big)^{\frac{1}{2}} \\ &= (\sqrt[4]{t})^{-3-\frac{D}{2}}\Big(\int(\partial_1(-\partial_2-\partial_1^2)\psi_1)^2\Big)^{\frac{1}{2}} \sim t^{-\frac{9}{8}}. \end{aligned} \tag{14}$$

Hence (13), evaluated in a point y, diverges w. r. t. $\mathbb{E}^{\frac{1}{2}}|\cdot|^2$ – while it converges as a Schwartz distribution.

In this sense we have $\mathbb{E}(\partial_1 v(y))^2 = +\infty$; in view of (14), this divergence arises from $t \downarrow 0$, that is, from small space/time scales, and thus is called an ultra-violet (UV) divergence. A quick fix is to introduce an UV cut-off, which for instance can be implemented by mollifying ξ. Using the semi-group convolution ξ_τ specifies the UV cut-off scale to be of the order of $\sqrt[4]{\tau}$. It is easy to check that in this case

$$\mathbb{E}(\partial_1 v(y))^2 \sim (\sqrt[4]{\tau})^{-\frac{1}{2}}.$$

The goal is to modify the equation (1) by "counter terms" such that

- the solution manifold stays under control as the ultra-violet cut-off $\tau \downarrow 0$.
- invariances of the solution manifold are preserved

In view of the above discussion, we expect the coefficients of the counter terms to diverge as the cut-off tends to zero.

sec:post

3. Postulates on the form of the counter terms

In view of $\alpha \in (0,1)$, u is a function while we think of all derivatives $\partial^{\mathbf{n}}u$ as being only Schwartz distributions. Hence it is natural to start from the very general Ansatz that the counter term is a polynomial in $\{\partial^{\mathbf{n}}u\}_{\mathbf{n}\neq\mathbf{0}}$ with coefficients that are general (local) functions in u:

ao23

$$(\partial_2-\partial_1^2)u + \sum_\beta h_\beta(u)\prod_{\mathbf{n}\neq\mathbf{0}}(\partial^{\mathbf{n}}u)^{\beta(\mathbf{n})} = a(u)\partial_1^2 u + \xi, \tag{15}$$

where β runs over all multi-indices[3] in $\mathbf{n}\neq\mathbf{0}$.

Only counter terms that have an order strictly below the order of the leading $\partial_2 - \partial_1^2$ are desirable, so that one postulates that the sum in (15) restricts to those multi-indices for which

cw14

$$|\beta|_p := \sum_{\mathbf{n}\neq\mathbf{0}}|\mathbf{n}|\beta(\mathbf{n}) < 2. \tag{16}$$

[3] which associate to every index $\mathbf{n}$ a $\beta(\mathbf{n}) \in \mathbb{N}_0$ such that $\beta(\mathbf{n})$ vanishes for all but finitely many $\mathbf{n}$'s

This leaves only $\beta = 0$ and $\beta = e_{(1,0)}$, where the latter means $\beta(\mathbf{n}) = \delta_{\mathbf{n}}^{(1,0)}$, so that (15) collapses to

ao24 $$(\partial_2 - \partial_1^2)u + h(u) + h'(u)\partial_1 u = a(u)\partial_1^2 u + \xi. \tag{17}$$

One also postulates that h and h' depend on the noise ξ only through its law / distribution / ensemble, hence are deterministic. Since we assume that the law is invariant under space-time translation, i. e. is stationary, it was natural to postulate that h and h' do not explicitly depend on x, hence are homogeneous.

Reflection symmetry. Let us now assume that

ao30bis $$\begin{aligned} &\text{the law of } \xi \text{ is invariant under space-time translation } y \mapsto y + x \\ &\text{and space reflection } y \mapsto (-y_1, y_2). \end{aligned} \tag{18}$$

We now argue that under this assumption, it is natural to postulate that the term $h'(u)\partial_1 u$ in (17) is not present, so that we are left with

ao27 $$(\partial_2 - \partial_1^2)u + h(u) = a(u)\partial_1^2 u + \xi. \tag{19}$$

To this purpose, let $x \in \mathbb{R}^2$ be arbitrary yet fixed, and consider the reflection at the line $\{y_1 = x_1\}$ given by $Ry = (2x_1 - y_1, y_2)$, which by pull back acts on functions as $\tilde{u}(y) = u(Ry)$. Since (1) features no explicit y-dependence, and only involves even powers of ∂_1, which like ∂_2 commute with R, we have

ao29 $$(u, \xi) \text{ satisfies (1)} \implies (\tilde{u}, \tilde{\xi}) \text{ satisfies (1)}. \tag{20}$$

Since we postulated that h and h' depend on ξ only via its law, and since in view of the assumption (18), $\tilde{\xi}$ has the same law as ξ, it is natural to postulate that the symmetry (20) extends from (1) to (17). Spelled out, this means that (17) implies

$$(\partial_2 - \partial_1^2)\tilde{u} + h(\tilde{u}) + h'(\tilde{u})\partial_1 \tilde{u} = a(\tilde{u})\partial_2 \tilde{u} + \tilde{\xi}.$$

Evaluating both identities at $y = x$, and taking the difference, we get for any solution of (17) that $h'(u(x))\partial_1 u(x) = h'(u(x))(-\partial_1 u(x))$, and thus $h'(u(x))\partial_1 u(x) = 0$, as desired.

Covariance under u-shift. We now come to our most crucial postulate, which restricts how the nonlinearity h depends on the nonlinearity / constitutive law a. Hence we no longer think of a single nonlinearity a, but consider all non-linearities at once, in the spirit of rough paths. This point of view reveals another invariance of (1), namely for any shift $v \in \mathbb{R}$

ao32 $$(u, a) \text{ satisfies (1)} \implies (u - v, a(\cdot + v)) \text{ satisfies (1)}. \tag{21}$$

A priori, h is a function of the u-variable that has a functional dependence on a, as denoted by $h = h[a](u)$. We postulate that the symmetry (21) extends from (1) to (19). This is the case under the following shift-covariance property

ao04 (22) $$h[a](u+v) = h[a(\cdot + v)](u) \quad \text{for all } u \in \mathbb{R}.$$

This property can also be paraphrased as: Whatever algorithm one uses to construct h from a, it should not depend on the choice of origin in what is just an affine space $\mathbb{R} \ni u$. Property (22) implies that the counter term is determined by a functional $c = c[a]$ on the space of nonlinearities a:

ao09 (23) $$h[a](v) = c[a(\cdot + v)].$$

Renormalization now amounts to choosing c such that the solution manifold stays under control as the UV regularization of ξ tends to zero.

ss:3.1

4. Algebrizing the counter term

In this section, we algebrize the relationship between a and the counter term h given by a functional c as in (23). To this purpose, we introduce the following coordinates on the space of analytic functions a of the variable u:

ao11 (24) $$\mathsf{z}_k[a] := \frac{1}{k!}\frac{d^k a}{du^k}(0) \quad \text{for } k \geq 0.$$

These are made such that by Taylor's

ao02 (25) $$a(u) = \sum_{k\geq 0} u^k \mathsf{z}_k[a] \quad \text{for } a \in \mathbb{R}[u],$$

where $\mathbb{R}[u]$ denotes the algebra of polynomials in the single variable u with coefficients in $\mathbb{R}$.

We momentarily specify to functionals c on the space of analytic a's that can be represented as polynomials in the (infinitely many) variables z_k. This leads us to consider the algebra $\mathbb{R}[\mathsf{z}_k]$ of polynomials in the variables z_k with coefficients in $\mathbb{R}$. The monomials

ao14 (26) $$\mathsf{z}^\beta := \prod_{k\geq 0} \mathsf{z}_k^{\beta(k)}$$

form a basis of this (infinite dimensional) linear space, where β runs over all multi-indices[4]. Hence as a linear space, $\mathbb{R}[\mathsf{z}_k]$ can be seen as the direct sum over the index set given by all multi-indices β, and we think of c as being of the form

ao16 (27) $$c[a] = \sum_{\beta} c_\beta \mathsf{z}^\beta[a] \quad \text{for } c \in \mathbb{R}[\mathsf{z}_k].$$

[4]which means they associate a frequency $\beta(k) \in \mathbb{N}_0$ to every $k \geq 0$ such that all but finitely many $\beta(k)$'s vanish

INFINITESIMAL u-SHIFT. Given a shift $v \in \mathbb{R}$, we start from $\mathbb{R} \ni u \mapsto u + v \in \mathbb{R}$, which by pull back leads to $a \mapsto a(\cdot + v)$; this provides an action/representation of the group $\mathbb{R}$ on $\mathbb{R}[u]$. Note that for $c \in \mathbb{R}[\mathsf{z}_k]$ and $a \in \mathbb{R}[u]$, the function $\mathbb{R} \ni v \mapsto c[a(\cdot + v)] = \sum_\beta c_\beta \prod_{k\geq 0}(\frac{1}{k!}\frac{d^k a}{du}(v))^{\beta(k)}$ is polynomial. Thus

ao06
$$(D^{(\mathbf{0})}c)[a] = \frac{d}{dv}_{|v=0} c[a(\cdot + v)] \tag{28}$$

is well-defined, linear in c and even a derivation in c, meaning that Leibniz's rule holds

ao15
$$(D^{(\mathbf{0})}cc') = (D^{(\mathbf{0})}c)c' + c(D^{(\mathbf{0})}c'). \tag{29}$$

The latter implies that $D^{(\mathbf{0})}$ is determined by its value on the coordinates z_k, which by definitions (24) and (28) is given by $D^{(\mathbf{0})}\mathsf{z}_k = (k+1)\mathsf{z}_{k+1}$. Hence $D^{(\mathbf{0})}$ has to agree with the derivation on the algebra $\mathbb{R}[\mathsf{z}_k]$

ao13
$$D^{(\mathbf{0})} = \sum_{k\geq 0}(k+1)\mathsf{z}_{k+1}\partial_{\mathsf{z}_k}, \tag{30}$$

which is well defined since the sum is effectively finite when applied to a monomial.

REPRESENTATION OF COUNTER TERM. Iterating (28) we obtain by induction in $l \geq 0$ for $c \in \mathbb{R}[\mathsf{z}_k]$ and $a \in \mathbb{R}[u]$

$$\frac{d^l}{dv^l}_{|v=0} c[a(\cdot + v)] = ((D^{(\mathbf{0})})^l c)[a]$$

and thus by Taylor's (recall that $v \mapsto c[a(\cdot + v)]$ is polynomial)

ao07
$$c[a(\cdot + v)] = \big(\sum_{l\geq 0}\frac{1}{l!}v^l(D^{(\mathbf{0})})^l c\big)[a]. \tag{31}$$

We combine (31) with (23) to

cw11
$$h[a](v) = \big(\sum_{l\geq 0}\frac{1}{l!}v^l(D^{(\mathbf{0})})^l c\big)[a]. \tag{32}$$

Hence our goal is to determine the coefficients c_β, which typically will blow up as $\tau \downarrow 0$.

ss:3.2

5. THE CENTERED MODEL

The purpose of this section is to motivate the notion of a centered model; the motivation will be in parts formal.

PARAMETERIZATION OF THE SOLUTION MANIFOLD. If $a \equiv 0$ it follows from (22) that h is a (deterministic) constant. We learned from the discussion after Lemma 1 that – given a base point x – there is a distinguished solution v (with $v(x) = 0$). Hence we may *canonically* parameterize a general solution u of (19) via $u = v + p$, by space-time functions p with $(\partial_2 - \partial_1^2)p = 0$. Such p are necessarily analytic. Having realized this, it is convenient[5] to free oneself from the constraint $(\partial_2 - \partial_1^2)p = 0$, which can be done at the expense of relaxing (19) to

ao43 (33) $$(\partial_2 - \partial_1^2)v = \xi + q \quad \text{for some analytic space-time function } q.$$

Since we think of ξ as being rough while q is infinitely smooth, this relaxation is still constraining v.

The implicit function theorem suggests that this parameterization (locally) persists in the presence of a sufficiently small analytic nonlinearity a: The nonlinear manifold of all space-time functions u that satisfy

ao45 (34) $$(\partial_2 - \partial_1^2)u + h(u) = a(u)\partial_1^2 u + \xi + q \quad \text{for some analytic space-time function } q$$

is parameterized by space-time analytic functions p. We now return to the point of view of Section 3 of considering all nonlinearities a at once, meaning that we consider the (still nonlinear) space of all space-time functions that satisfy (34) for *some* analytic nonlinearity a. We want to capitalize on the symmetry (21), which extends from (1) to (19) and to (34). We do so by considering the above space of u's *modulo constants*, which we implement by focusing on increments $u - u(x)$. Summing up, it is reasonable to expect that the space of all space-time functions u, modulo space-time constants, that satisfy (34) for some analytic nonlinearity a and space-time function q (but at fixed ξ), is parameterized by pairs (a, p) with $p(x) = 0$.

Formal series representation. In line with the term-by-term approach from physics, we write $u(y) - u(x)$ as a (typically divergent) power series

ao83 (35) $$u(y) - u(x) = \sum_\beta \Pi_{x\beta}(y) \prod_{k\ge 0} \Big(\frac{1}{k!}\frac{d^k a}{du^k}(u(x))\Big)^{\beta(k)} \prod_{\mathbf{n}\neq \mathbf{0}} \Big(\frac{1}{\mathbf{n}!}\partial^{\mathbf{n}} p(x)\Big)^{\beta(\mathbf{n})},$$

where β runs over all multi-indices in $k \ge 0$ and $\mathbf{n} \neq \mathbf{0}$, and where $\mathbf{n}! := (n_1!)(n_2!)$. Introducing coordinates on the space of analytic space-time functions p with $p(0) = 0$ via

ao48 (36) $$\mathsf{z}_{\mathbf{n}}[p] = \frac{1}{\mathbf{n}!}\partial^{\mathbf{n}} p(0) \quad \text{for } \mathbf{n} \neq \mathbf{0},$$

(35) can be more compactly written as

ao01 (37) $$u(y) = u(x) + \sum_\beta \Pi_{x\beta}(y)\mathsf{z}^\beta[a(\cdot + u(x)), p(\cdot + x) - p(x)].$$

[5] otherwise, the coordinates $\mathsf{z}_{(2,0)}$ and $\mathsf{z}_{(0,1)}$ defined in (36) would be redundant on p-space

This is reminiscent of Butcher series in the analysis of ODE discretizations.

Recall from above that for $a \equiv 0$ we have the explicit parameterization

cw13 $$u - u(x) = v + p \tag{38}$$

with the distinguished solution v of the linear equation. Hence from setting $a \equiv 0$ and $p \equiv 0$ in (35), we learn that $\Pi_{x0} = v$. From keeping $a \equiv 0$ but letting p vary we then deduce that for all multi-indices $\beta \neq 0$ which satisfy $\beta(k) = 0$ for all $k \geq 0$ we must have[6]

ao59 $$\Pi_{x\beta}(y) = \left\{ \begin{array}{cc} (y-x)^{\mathbf{n}} & \text{provided } \beta = e_{\mathbf{n}} \\ 0 & \text{else} \end{array} \right\}. \tag{39}$$

Hierarchy of linear equations. The collection of coefficients $\{\Pi_{x\beta}(y)\}_\beta$ from (37) is an element of the direct *product* with the same index set as the direct *sum* $\mathbb{R}[\mathsf{z}_k, \mathsf{z}_{\mathbf{n}}]$. Hence the direct product inherits the multiplication of the polynomial algebra

ao52 $$(\pi\pi')_{\bar\beta} = \sum_{\beta+\beta'=\bar\beta} \pi_\beta \pi'_{\beta'}, \tag{40}$$

and is denoted as the (well-defined) algebra $\mathbb{R}[[\mathsf{z}_k, \mathsf{z}_{\mathbf{n}}]]$ of formal power series; we denote by $\mathsf{1}$ its unit element. We claim that in terms of (37), (34) assumes the form of

cw09 $$(\partial_2 - \partial_1^2)\Pi_x = \Pi_x^- \quad \text{up to space-time analytic functions} \tag{41}$$

where

ao49 $$\Pi_x^- := \sum_{k\geq 0} \mathsf{z}_k \Pi_x^k \partial_1^2 \Pi_x - \sum_{l\geq 0} \frac{1}{l!} \Pi_x^l (D^{(\mathbf{0})})^l c + \xi_\tau \mathsf{1}, \tag{42}$$

as an identity in formal power series in $\mathsf{z}_k, \mathsf{z}_{\mathbf{n}}$ with coefficients that are continuous space-time functions. We shall argue below that (42) is effectively, i. e. componentwise, well-defined despite the two infinite sums, and despite extending from $c \in \mathbb{R}[\mathsf{z}_k]$ to $c \in \mathbb{R}[[\mathsf{z}_k]]$.

Here comes the formal argument that relates $\{\partial_2, \partial_1^2\}u$, $a(u)$, and $h(u)$, to $\{\partial_2, \partial_1^2\}\Pi_x[\tilde a, \tilde p]$, $\sum_{k\geq 0} \mathsf{z}_k \Pi_x^k[\tilde a, \tilde p]$, and $\sum_{l\geq 0} \frac{1}{l!}\Pi_x^l (D^{(\mathbf{0})})^l c[\tilde a, \tilde p]$, respectively. Here we have set for abbreviation $\tilde a = a(\cdot + u(x))$ and $\tilde p = p(\cdot + x) - p(x)$. It is based on (37), which can be compactly written as $u(y) = u(x) + \Pi_x[\tilde a, \tilde p](y)$. Hence the statement on $\{\partial_2, \partial_1^2\}u$ follows immediately. Together with $a(u(y)) = \tilde a(u(y) - u(x))$, this also implies by (25) the desired

$$a(u(y)) = \big(\sum_{k\geq 0} \mathsf{z}_k \Pi_x^k(y)\big)[\tilde a, \tilde p].$$

[6] where $\beta = e_{\mathbf{n}}$ denotes the multi-index with $\beta(\mathbf{m}) = \delta_{\mathbf{m}}^{\mathbf{n}}$ next to $\beta(k) = 0$

Likewise by (22), we have $h[a](u(y)) = h[\tilde a](u(y) - u(x))]$, so that by (32), we obtain the desired

$$h[a](u(y)) = \big(\sum_{l\ge 0}\frac{1}{l!}\Pi_x^l(y)(D^{(\mathbf{0})})^l c\big)[\tilde a, \tilde p].$$

Finiteness properties. The next lemma collects crucial algebraic properties.

lem:alg **Lemma 2.** *The derivation $D^{(\mathbf{0})}$ extends from $\mathbb{R}[\mathsf{z}_k]$ to $\mathbb{R}[[\mathsf{z}_k]]$.*

Moreover, for $\pi, \pi' \in \mathbb{R}[[\mathsf{z}_k, \mathsf{z}_{\mathbf{n}}]]$, $c \in \mathbb{R}[[\mathsf{z}_k]]$, and $\xi \in \mathbb{R}$,

cw07 (43)
$$\pi^- := \sum_{k\ge 0}\mathsf{z}_k \pi^k \pi' - \sum_{l\ge 0}\frac{1}{l!}\pi^l (D^{(\mathbf{0})})^l c + \xi 1 \ \in\ \mathbb{R}[[\mathsf{z}_k, \mathsf{z}_{\mathbf{n}}]]$$

are well-defined, in the sense that the sums are componentwise finite.

Finally, for

cw15 (44)
$$[\beta] := \sum_{k\ge 0} k\beta(k) - \sum_{\mathbf{n}\neq\mathbf{0}}\beta(\mathbf{n})$$

we have the implication

$$\pi_\beta = \pi'_\beta = 0 \quad \text{unless} \quad [\beta]\ge 0 \text{ or } \beta = e_{\mathbf{n}} \text{ for some } \mathbf{n}\neq\mathbf{0}$$
$$\Longrightarrow$$

cw08 (45)
$$\pi^-_\beta = 0 \quad \text{unless} \quad \left\{\begin{array}{l} [\beta]\ge 0 \text{ or} \\ \beta = e_k + e_{\mathbf{n}_1} + \cdots + e_{\mathbf{n}_k} \\ \text{for some } k\ge 1 \text{ and } \mathbf{n}_1, \cdots, \mathbf{n}_k \neq \mathbf{0}. \end{array}\right\}.$$

We note that for β as in the second alternative on the r. h. s. of (45), it follows from (39) that $\Pi^-_{x\beta}$ is a polynomial. Hence in view of the modulo in (41), we learn from (45) that we may assume

cw03 (46)
$$\Pi_{x\beta}\equiv 0 \quad \text{unless} \quad [\beta]\ge 0 \ \text{ or } \ \beta = e_{\mathbf{n}} \text{ for some } \mathbf{n}\neq\mathbf{0}.$$

Proof of Lemma 2. We first address the extension of $D^{(\mathbf{0})}$ and note that from (30) we may read off the matrix representation of $D^{(\mathbf{0})}$ $\in \mathrm{End}(\mathbb{R}[\mathsf{z}_k])$ w. r. t. (26) given by

$$(D^{(\mathbf{0})})_\beta^\gamma = (D^{(\mathbf{0})}\mathsf{z}^\gamma)_\beta \overset{(30)}{=} \sum_{k\ge 0}(k+1)\big(\mathsf{z}_{k+1}\partial_{\mathsf{z}_k}\mathsf{z}^\gamma\big)_\beta$$

ao20 (47)
$$\overset{(26)}{=} \sum_{k\ge 0}(k+1)\gamma(k)\left\{\begin{array}{ll} 1 & \text{provided } \gamma + e_{k+1} = \beta + e_k \\ 0 & \text{otherwise} \end{array}\right\}.$$

From this we read off that $\{\gamma | (D^{(\mathbf{0})})^\gamma_\beta \neq 0\}$ is finite for every β, which implies that $D^{(\mathbf{0})}$ extends from $\mathbb{R}[\mathsf{z}_k]$ to $\mathbb{R}[[\mathsf{z}_k]]$. With help of (40) the derivation property (29) can be expressed coordinate-wise, and thus extends to $\mathbb{R}[\mathsf{z}_k]$.

We now turn to (43), which component-wise reads

ao51 (48)
$$\pi^-_\beta = \sum_{k\geq 0} \sum_{e_k+\beta_1+\cdots+\beta_{k+1}=\beta} \pi_{\beta_1}\cdots\pi_{\beta_k}\pi'_{\beta_{k+1}} - \sum_{l\geq 0}\frac{1}{l!}\sum_{\beta_1+\cdots+\beta_{k+1}=\beta} \pi_{\beta_1}\cdots\pi_{\beta_k}((D^{(\mathbf{0})})^l c)_{\beta_{k+1}} + \xi\delta^0_\beta, \tag{48}$$

and claim that the two sums are effectively finite. For the first r. h. s. this is obvious since thanks to the presence of[7] e_k in $e_k+\beta_1+\cdots+\beta_{k+1}=\beta$, for fixed β there are only finitely many $k\geq 0$ for which this relation can be satisfied.

In preparation for the second r. h. s. term of (48) we now establish that

ao19 (49)
$$((D^{(\mathbf{0})})^l)^\gamma_\beta = 0 \quad \text{unless} \quad [\beta]_0 = [\gamma]_0 + l, \tag{49}$$

where we (momentarily) introduced the scaled length $[\gamma]_0 := \sum_{k\geq 0} k\gamma(k) \in \mathbb{N}_0$. The argument for (49) proceeds by induction in $l\geq 0$. It is tautological for the base case $l=0$. In order to pass from l to $l+1$ we write $((D^{(\mathbf{0})})^{l+1})^\gamma_\beta = \sum_{\beta'}((D^{(\mathbf{0})})^l)^{\beta'}_\beta (D^{(\mathbf{0})})^\gamma_{\beta'}$; by induction hypothesis, the first factor vanishes unless $[\beta]_0 = [\beta']_0 + l$. We read off (47) that the second factor vanishes unless $[\beta']_0 = [\gamma]_0 + 1$, so that the product vanishes unless $[\beta]_0 = [\gamma]_0 + (l+1)$, as desired.

Equipped with (49) we now turn to the second r. h. s. term of (48) and note that $((D^{(\mathbf{0})})^l c)_{\beta_{k+1}}$ vanishes unless $l \leq [\beta_{k+1}]_0 \leq [\beta]_0$, which shows that also here, only finitely many $l\geq 0$ contribute for fixed β.

Homogeneity. The homogeneity $|\beta|$ of a multi-index β is motivated by a scaling invariance in law of the manifold of solutions to (22): We start with a parabolic rescaling of space and time according to $x_1 = \lambda\hat{x}_1$ and $x_2 = \lambda^2\hat{x}_2$. Our assumption on the noise ensemble is consistent with[8] $\xi =_{\rm law} \lambda^{\alpha-2}\hat{\lambda}$. This translates into the desired $u =_{\rm law} \lambda^\alpha\hat{u}$, provided we transform the nonlinearities according to $a(u) = \hat{a}(\lambda^{-\alpha}u)$ and $h(u) = \lambda^{\alpha-2}\hat{h}(\lambda^{-\alpha}u)$. On the level of the coordinates (25) the former translates into $\mathsf{z}_k = \lambda^{-\alpha k}\hat{\mathsf{z}}_k$. When it comes to the parameter p it is consistent with (38) that it scales like u, i. e. $p = \lambda^\alpha\hat{p}$, so that the coordinates (36) transform according to $\mathsf{z}_{\mathbf{n}} = \lambda^{\alpha-|\mathbf{n}|}\hat{\mathsf{z}}_{\mathbf{n}}$. Hence we read off (37) that $\Pi_{x\beta} =_{\rm law} \lambda^{|\beta|}\hat{\Pi}_{\hat{x}\beta}$, where

$$|\beta| := \alpha(1+[\beta]) + |\beta|_p,$$

recalling the definitions (16) and (44).

[7] $\gamma = e_k$ denotes the multi-index with $\gamma(l) = \delta^k_l$ next to $\gamma(\mathbf{n}) = 0$

[8] which for $\alpha = \frac{1}{2}$ turns into the well-known invariance of white noise

6. The main result

Theorem 1. *Suppose the law of ξ is invariant under translation and spatial reflection; suppose that it satisfies a spectral gap inequality with exponent $\alpha \in (1-\frac{D}{4}, 1)$ with $\alpha \notin \mathbb{Q}$.*

Then given $\tau > 0$, there exists a deterministic $c \in \mathbb{R}[[\mathsf{z}_k]]$, and for every $x \in \mathbb{R}^2$, a random[9] *$\Pi_x \in C^2[[\mathsf{z}_k, \mathsf{z}_\mathbf{n}]]$, and a random $\Pi_x^- \in C^0[[\mathsf{z}_k, \mathsf{z}_\mathbf{n}]]$ that are related by (42) and*

cw04
$$(\partial_2 - \partial_1^2)\Pi_{x\beta} = \Pi_{x\beta}^- + \text{polynomial of degree} \leq |\beta| - 2, \tag{50}$$

and that satisfy (39), the population condition (46) and

cw10
$$c_\beta \quad \text{unless} \quad |\beta| \geq 2. \tag{51}$$

Moreover, we have the estimates

cw01
$$\mathbb{E}^{\frac{1}{p}}|\Pi_{x\beta}(y)|^p \lesssim_{\beta,p} |y-x|^{|\beta|}, \tag{52}$$

cw02
$$\mathbb{E}^{\frac{1}{p}}|\Pi_{x\beta t}^-(y)|^p \lesssim_{\beta,p} (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t} + |y-x|)^{|\beta|-\alpha}. \tag{53}$$

As we aimed for, estimate (52) establishes control of the solution manifold, at least in the term-by-term fashion via (35), that is uniform in the UV cut-off $\tau \downarrow 0$.

We remark that we may pass from (53) to (52) by Lemma 1. Indeed, because of (46) we may restrict to β with $[\beta] \geq 0$. In this case, by $\alpha \notin \mathbb{Q}$, $|\beta| = \alpha(1 + [\beta]) + |\beta|_p \notin \mathbb{Z}$, next to $|\beta| \geq \alpha$. Hence we may indeed apply Lemma 1 with $\eta = |\beta|$ and (53) as input. The output yields a (unique) $\Pi_{x\beta}$ satisfying (50) and (52).

We further remark that the counter term c is implicitly determined.

7. The spectral gap (SG) condition

[9] by this we mean a formal power series in $\mathsf{z}_k, \mathsf{z}_\mathbf{n}$ with values in the twice continuously differentiable space-time functions